Solution for HW5

1-11-2016

§38) 3) When $m=n$,

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta}\, d\theta = \int_0^{2\pi} d\theta = 2\pi.$$

When $m \neq n$,

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta}\, d\theta = \int_0^{2\pi} e^{i(m-n)\theta}\, d\theta$$

$$= \left.\frac{e^{i(m-n)\theta}}{i(m-n)}\right|_0^{2\pi}$$

$$= \frac{1}{i(m-n)}\left(e^{i(m-n)(2\pi)} - 1\right)$$

$$= 0$$

4). Since

$$\int_0^{\pi} e^{(1+i)x}\, dx = \left.\frac{e^{(1+i)x}}{1+i}\right|_0^{\pi}$$

$$= \frac{1}{1+i}\left(e^{\pi+i\pi} - 1\right)$$

$$= \frac{1}{1+i}\left(-e^{\pi} - 1\right)$$

$$= \frac{1-i}{2}\left(-e^{\pi} - 1\right)$$

$$= -\frac{(1+e^{\pi})}{2} + i\,\frac{(1+e^{\pi})}{2},$$

By comparing the real and imaginary parts on both sides, we have

$$\int_0^{\pi} e^{x}\cos x\, dx = -\frac{(1+e^{\pi})}{2} \quad \text{and}$$

$$\int_0^{\pi} e^{x}\sin x\, dx = \frac{(1+e^{\pi})}{2}$$

§42) 1) $f(z) = \frac{z+2}{z} = 1 + \frac{2}{z}$

a) $\int_C f(z)\,dz = \int_0^{\pi} \left(1 + \frac{2}{2e^{i\theta}}\right)(2ie^{i\theta})\,d\theta$

$= \int_0^{\pi} (2ie^{i\theta} + 2i)\,d\theta$

$= 2e^{i\theta} + 2i\theta \Big|_0^{\pi}$

$= -4 + 2\pi i$

b) $\int_C f(z)\,dz = \int_{\pi}^{2\pi} \left(1 + \frac{2}{2e^{i\theta}}\right)(2ie^{i\theta})\,d\theta$

$= 2e^{i\theta} + 2i\theta \Big|_{\pi}^{2\pi}$

$= 4 + 2\pi i$

c) By a) and b)

$\int_C f(z)\,dz = \int_0^{2\pi} \left(1 + \frac{2}{2e^{i\theta}}\right)(2ie^{i\theta})\,d\theta$

$= 2e^{i\theta} + 2i\theta \Big|_0^{2\pi}$

$= 4\pi i.$

4) $\int_C f(z)\,dz = \int_{-1}^{0} (1)(1+3x^2 i)\,dx + \int_0^1 (4x^3)(1+3x^2 i)\,dx$

$= (x + x^3 i)\Big|_{-1}^{0} + (x^4 + 2x^6 i)\Big|_0^1$

$= (0 - (-1-i)) + (1+2i)$

$= 2+3i.$

7) $\int_C f(z)\,dz = \int_0^{\pi} e^{i\,\mathrm{Log}(e^{i\theta})} \cdot ie^{i\theta}\,d\theta$

$= \int_0^{\pi} e^{i(\ln|e^{i\theta}| + i\theta)} \cdot ie^{i\theta}\,d\theta$

$= i\int_0^{\pi} e^{(-1+i)\theta}\,d\theta$

$= \frac{i}{-1+i} e^{(-1+i)\theta}\Big|_0^{\pi}$

$= \frac{i}{-1+i}\left(e^{-\pi+i\pi} - 1\right)$

$= \frac{i(-1-i)}{2}\left(-e^{-\pi} - 1\right)$

$= -\frac{1+e^{-\pi}}{2}(1-i)$

§43) 1) $\because \left|\frac{1}{z^2-1}\right| \le \frac{1}{|z|^2-1} = \frac{1}{2^2-1} = \frac{1}{3}$.

$\therefore \left|\int_C \frac{dz}{z^2-1}\right| \le \frac{1}{3}(\text{length of } C) = \frac{\pi}{3}$

4) Note that (1) $|2z^2-1| \le 2|z|^2+1 = 2R^2+1$

(3) $|z^4+5z^2+4| = |z^2+1||z^2+4|$

$\ge (|z|^2-1)(|z|^2-4)$

$= (R^2-1)(R^2-4)$

So $\left|\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}\,dz\right| \le \frac{2R^2+1}{(R^2-1)(R^2-4)}(\text{length of } C)$

$= \frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)}$

$\therefore \lim_{R\to\infty} \frac{\pi R(R^2-1)}{(R^2-1)(R^2-4)} = \lim_{R\to\infty} \frac{\frac{\pi}{R}(1-\frac{1}{R^2})}{(1-\frac{1}{R^2})(1-\frac{4}{R^2})} = 0$

$$\therefore \lim_{R\to\infty} \left| \int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}\,dz \right| = 0$$

§45) 1) $\int_C z^n\,dz = \left.\frac{z^{n+1}}{n+1}\right|_{z_1}^{z_2} = \frac{1}{n+1}\left(z_2^{n+1} - z_1^{n+1}\right)$

2) b) $\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz = \left. 2\sin\left(\frac{z}{2}\right)\right|_0^{\pi+2i}$

$$= 2\sin\left(\frac{\pi}{2}+i\right)$$

$$= \frac{e^{i\left(\frac{\pi}{2}+i\right)} - e^{-i\left(\frac{\pi}{2}+i\right)}}{i}$$

$$= e^{-1} + e$$

c) $\int_1^3 (z-2)^3\,dz = \left.\frac{(z-2)^4}{4}\right|_1^3 = 0$

5) $\int_{-1}^1 z^i\,dz = \left.\frac{z^{i+1}}{i+1}\right|_{-1}^1$

$$= \frac{1}{1+i}\left(e^{(1+i)\log 1} - e^{(1+i)\log(-1)}\right)$$

$$= \frac{1}{1+i}\left(e^{(1+i)(\ln|1| + i(0))} - e^{(1+i)(\ln|-1| + i(\pi))}\right)$$

$$= \frac{1}{1+i}\left(1 - e^{i\pi - \pi}\right)$$

$$= \frac{1+e^{-\pi}}{2}(1-i)$$